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A Procedure for Improving Discrete Substructure Representation in Dynamic Synthesis

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In a discrete substructure synthesis method developed by the authors, the motion of each substructure is represented by a given number of shape vectors called "admissible vectors." To force the individual substructures to act together so as to form a whole structure, approximate geometric compatibility conditions are imposed by means of an approach based on weighted residuals. A structure defined by the approximate compatibility conditions is referred to as an "intermediate structure." This paper develops a general iterative procedure for improving the admissible vectors representing each discrete substructure in the synthesis. The procedure permits the computation of an improved eigensolution for the intermediate structure, without increasing the number of degrees of freedom used to represent each substructure. In any iteration, the computations associated with each substructure are independent of those for all other substructures. Hence, they can be performed in parallel. By increasing the number of iterations, the eigensolution for the intermediate structure is approached. Numerical results indicate that convergence is very rapid. Indeed, one or two iterations may be sufficient for many problems.

I. Introduction

An important problem in the dynamic analysis for flexible structures is that of determining the natural frequencies and associated natural modes of vibration, particularly the lower ones. Complex structures are often represented by mathematical models, such as finite-element models, possessing a very large number of degrees of freedom, where the number can reach into the tens of thousands. Because of the large number of degrees of freedom, a direct iterative method for obtaining a partial eigensolution, know as subspace iteration, was developed. The subspace iteration method has a long history. An extensive bibliography is given in Ref. 1 and the current status of the method is discussed in Refs. 2-4 in the context of structural dynamics. Even the subspace iteration method, however, can be overwhelmed by the large number of degrees of freedom in the mathematical model, so that a method for prior reduction of the number of degrees of freedom is desirable.

Complex structures are often modeled by breaking the structure into a number of simpler components or substructures. The substructure models are then coupled together to form the whole structure model. The technique is known as substructure synthesis and its origin can be found in Refs. 5 and 6. The idea of Refs. 5 and 6 is to represent the motion of each substructure by a set of substructure normal modes, obtained by solving an eigenvalue problem for each substructure. Then each substructure is represented in the synthesis process by a reduced number of lower substructure modes. The synthesis leads to an eigenvalue problem for the assembled structure of a substantially smaller order than that of the original formulation. The price paid in reducing the order of the eigenvalue problem is that its solution is only an approximation of the actual eigensolution of the original structure.

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It must be recognized that substructure eigenvalue problems cannot be defined uniquely, so that there are various types of substructure modes. In fact, the type of substructure modes used affects the accuracy of the computed eigensolution. Hence, there have been many suggestions as to the type of substructure modes to be used. Substructure eigenvalue problems can be defined in which the boundary conditions at an internal boundary between adjacent substructures are specified as either fixed[5] or free[7-9] or to involve inertial and/or stiffness loadings.[10] It has also been suggested that supplementing substructure normal modes with sets of statically derived interface modes (e.g., constraint, attachment, and inertia-relief modes) can improve accuracy.[11-14] However, none of the methods presented in Refs. 5-14 can yield the exact eigensolution for the actual structure while using truncated sets of substructure modes. This fact raises questions as to the significance of using substructure modes.

For complicated substructures, producing substructure modes by solving a substructure eigenvalue problem is a computationally expensive task that is not really necessary.[15,16] Indeed, in a substructure synthesis method for discrete substructures developed recently by the authors,[17,18] the motion of each substructure is represented by a given number of admissible vectors. The concept of admissible vectors was first advanced by the authors in Ref. 19. Admissible vectors represent the discrete counterpart of admissible functions for continuous substructures. They must satisfy geometric boundary conditions imposed on a substructure and they must have acceptable shapes, where the latter is the discrete counterpart of differentiability requirements. The advantage of substructure admissible vectors, as opposed to substructure modes, is that they are easy to obtain and attractive to work with computationally.

In addition to the problem of representing each substructure, there is the problem of coupling together the various substructures to form a whole structure. While various specific coupling procedures are presented in Refs. 5-14, Refs. 17 and 18 adopt a general procedure for connecting together the otherwise disjointed substructures. The procedure consists of imposing approximate geometric compatibility by means of the method of weighted residuals. A structure whose internal boundary conditions are only approximations to the actual ones is referred to as an "intermediate structure." The intermediate structure represents a mathematical concept defined by the type of weighting vectors used and their number. Basically all substructure synthesis methods, including those described in Refs. 5-14, replace the actual structure by an intermediate structure, although Refs. 5-14 may not refer to it as such.

The substructure synthesis method produces a computed eigensolution approximating the actual eigensolution of the structure. As discussed in Refs. 17 and 18, the substructure synthesis method is a Rayleigh-Ritz method for an intermediate structure, so that the computed eigenvalues are upper bounds for the eigenvalues of the intermediate structure. The accuracy with which the computed eigensolution represents the eigensolution of the intermediate structure depends on the particular choice of substructure admissible vectors. To increase the accuracy, one can increase the number of admissible vectors used for each substructure. Another possibility, and one that is the subject of this paper, is to improve the sets of substructure admissible vectors systematically until the desired eigensolution accuracy is obtained, while using the same number of vectors in each set.

In this paper, an iterative procedure for improving the representation of each discrete substructure is developed. It is understood that discrete substructures are the result of discretizing an actual substructure, perhaps by the finite-element method, where the number of degrees of freedom of the substructure model is large. To develop the procedure, the concept of subspace iteration for a single discrete structure is presented first. Although the concept is well known, it is usually presented only in the context of positive definite structures. Both positive definite and positive semidefinite structures are considered in this paper. Next, the formulation for a single discrete substructure is considered, where the effects of adjacent substructures are represented as forces exerted on the substructure. Then, the discrete substructure synthesis method[17,18] is discussed. Finally, a subspace iteration procedure is presented in which the substructure synthesis is an integral part. A numerical example illustrating the very rapid convergence of the present procedure is given.

It must be emphasized that the procedure of this paper is a generalization of the concepts presented in Refs. 11-14 for improving the accuracy of component modes synthesis methods. In fact, existence of the procedure developed herein obviates finding substructure normal modes or considering such concepts as residual flexibility, inertia relief modes, constraint modes, and attachment modes.[11-14] At the same time, the procedure represents an extension of the concept of subspace iteration, developed previously for a single structure,[2-4] to structures composed of substructures. Therefore, the present procedure constitutes an important computational tool for computing a partial eigensolution for complex structures.

II. The Subspace Iteration Method: A Review

Our interest is in the eigenvalue problem for a structure represented by an n_T degree-of-freedom finite-element model. The eigenvalue problem is written in the matrix form

$$ku = \lambda m u \tag{1}$$

where k and m are $n_T \times n_T$ symmetric stiffness and mass matrices, respectively, and u is an n_T-dimensional configuration vector. The mass matrix is positive definite and the stiffness matrix is either positive definite or only positive semidefinite, depending on whether the structure is restrained or unrestrained. The vector u generally contains both translational and angular displacements.

The idea of subspace iteration[1-4] is to represent an approximate eigenvector u at iteration step p as the sum

$$u_p = \sum_{i=1}^{N} a_i \phi_i^p \tag{2}$$

of N linearly independent vectors ϕ_i^p multiplied by unknown coefficients a_i, where $N \ll n_T$. Then, an approximate eigensolution using the vectors ϕ_i^p is obtained by the Rayleigh-Ritz method, i.e., the coefficients a_i are determined so as to satisfy the Nth order eigenvalue problem

$$\sum_{j=1}^{N} [(\phi_i^p)^T k \phi_j^p - \Lambda^N (\phi_i^p)^T m \phi_j^p] a_j = 0, \quad i = 1,2,\ldots,N \tag{3}$$

The algebraic eigensolution consists of N eigenvalues Λ_r^N approximating the actual eigenvalues λ_r $(r = 1,2,\ldots,N)$, as well as coefficients $a_i^{(r)}$ which are used to compute approximate eigenvectors $u_p^{(r)}$ according to Eq. (2). The computed eigenvectors are orthogonal and they can be normalized so that $u_p^{(r)T} m u_p^{(s)} = \delta_{rs}$ $(r,s = 1,2,\ldots,N)$, where δ_{rs} is the Kronecker delta. The accuracy of the computed eigensolution can be increased by using a greater number of vectors ϕ_i^p or it can be increased by choosing a set of better vectors ϕ_i^{p+1}. Subspace iteration is concerned with the second alternative. When the stiffness matrix k is positive definite, improved vectors ϕ_i^{p+1} are produced by solving the n_T simultaneous algebraic equations.

$$k \phi_i^{p+1} = m u_p^{(i)}, \qquad i = 1,2,\ldots,N \tag{4}$$

The next iteration step, step $p+1$, consists of solving the algebraic eigenvalue problem (3) of order N obtained by using the improved vectors ϕ_i^{p+1}. The process of solving Eq. (4) is equivalent to operating on $u_p^{(i)}$ by matrix $A = k^{-1} m$ and it can be shown[1-4] that the effect is to decrease the relative magnitude of the contribution of each vector ϕ_i^{p+1} of a higher true eigenvector $u^{(r)}$ $(r = N+1, N+2,\ldots,n_T)$. Hence, the N-dimensional subspace E_{p+1}^N spanned by the vectors ϕ_i^{p+1} $(i = 1,2,\ldots,N)$ is closer to the N-dimensional subspace E^N spanned by the lowest N true eigenvectors $u^{(r)}$ and E_{p+1}^N approaches E^N as the number of iterations p becomes infinite.

When the stiffness matrix is positive semidefinite, and hence singular, the structure is unrestrained and the eigenvalue problem (1) admits a total of n_R eigenvectors $u^{(i)}$ $(i = 1,2,\ldots,n_R)$ corresponding to the repeated eigenvalue $\lambda = 0$. The eigenvectors $u^{(i)}$ $(i = 1,2,\ldots,n_R)$ are known as the rigid body modes of the structure and they satisfy $ku = 0$. Since the eigenvalue $\lambda = 0$ is repeated n_R times, any linear combination of the eigenvectors $u^{(i)}$ $(i = 1,2,\ldots,n_R)$ is also an eigenvector.[4] The difficulty with a singular stiffness matrix is that an inverse k^{-1} does not exist. To remove the singularity, we require that the configuration vector u satisfy the n_R independent constraint equations

$$Cu = 0 \tag{5}$$

where C is an $n_R \times n_T$ rectangular matrix. Although one possibility is to consider constraints that render u orthogonal to all n_R rigid body modes, simpler constraints are those that render n_R entries in the vector u corresponding to the translations and rotations of a single point equal to zero. In either case, the constraint equation (5) can be used to write a relation between an $(n_T - n_R)$-dimensional vector y of independent generalized coordinates and the n_T-dimensional constrained vector u_c in the form

$$u_c = C_c y \tag{6}$$

where C_c is an $n_T \times (n_T - n_R)$ rectangular matrix. A solution of Eq. (1) consists of the unique solution u_c relative to the constraints (5) plus a particular linear combination of the rigid body modes $u^{(i)}$ $(i = 1,2,\ldots,n_R)$ chosen to render the resulting vector orthogonal to all n_R rigid body modes.

For subspace iteration, when the stiffness matrix k is singular, the above discussion implies that an approximate eigenvector u_p at iteration step p must be represented as the sum

$$u_p = \sum_{i=1}^{N_R} a_i u^{(i)} + \sum_{j=1}^{N-n_R} a_{n_R+j}\phi_j^p, \qquad p=0,1,... \tag{7}$$

where a_i $(i=1,2,...,N)$ are unknown coefficients to be determined, $u^{(i)}$ $(i=1,2,...,n_R)$ are rigid body modes of the structure, and ϕ_j^p are known vectors linearly independent of all the rigid body modes. As before, the coefficients a_i are determined by solving the Nth order algebraic eigenvalue problem (3). Now the solution of the eigenvalue problem consists of n_R zero eigenvalues Λ_i^N $(i=1,2,...,n_R)$ corresponding to the n_R rigid body modes as well as $N-n_R$ computed eigenvalues $\Lambda_{n_R+j}^N$ approximating the lowest $N-n_R$ nonzero eigenvalues λ_{n_R+j} $(j=1,2,...,N-n_R)$. Associated with the computed lower nonzero eigenvalues are values of the coefficients $a_i^{(n_R+j)}$ $(i=1,2,...,N)$ that, when substituted into the sum (7), yield the corresponding computed eigenvectors $u_p^{(n_R+j)}$ $(j=1,2,...,N-n_R)$. The coefficients are always chosen so that the approximate eigenvectors $u_p^{(n_R+j)}$ $(j=1,2,...,N-n_R)$ are orthogonal to the rigid body modes. To increase the accuracy of the lower nonzero computed eigenvalues and eigenvectors, improved vectors ϕ_j^{p+1} are produced by first solving the n_T-n_R simultaneous algebraic equations

$$\bar{k}y_j^{p+1} = C_c^T m u_p^{(n_R+j)}, \qquad j=1,2,...,N-n_R \tag{8}$$

for y_j^{p+1} where $\bar{k}=C_c^T k C_c$ is an $(n_T-n_R)\times(n_T-n_R)$ positive definite symmetric matrix, and then substituting into

$$\phi_j^{p+1} = C_c y_j^{p+1} \tag{9}$$

The entire procedure can be repeated iteratively using the improved vectors ϕ_j^{p+1} in Eq. (7). Note that according to Eqs. (8) and (9), only the last $N-n_R$ vectors are changed in each iteration. This iterative process improves the vectors ϕ_j^p successively, until the N-dimensional subspace spanned by the rigid body modes and the vectors ϕ_j^p contains the first $N-n_R$ nonzero eigenvectors. Note that if the interest is in the lowest q *nonzero* eigenvalues and eigenvectors, we must take $N\geq q+n_R$.

We observe that subspace iteration requires the solution of an Nth order algebraic eigenvalue problem at each iteration. Because our interest is in obtaining only a relatively small number of eigenvalues and eigenvectors, the order N of the algebraic eigenvalue problem is sufficiently small that its solution is easy to obtain. This is in contrast to the original algebraic eigenvalue problem (1) of order n_T, in which n_T is so large that its solution is very difficult to obtain.

III. The Reciprocal Formulation of a Typical Substructure

The subspace iteration method is based on a "reciprocal" formulation for the whole structure, obtained by premultiplying Eq. (1) by k^{-1}. The purpose of this section is to present an analogous "reciprocal" formulation for a typical substructure s. To this end, the whole structure is assumed to be divided into m substructures. Each substructure s $(s=1,2,...,m)$ acts as part of the whole structure and the whole structure eigenvalue problem is described within substructure s by the algebraic equations

$$k_s u_s = \eta_s + \lambda m_s u_s \tag{10}$$

where k_s and m_s are $n_s \times n_s$ symmetric mass and stiffness matrices, respectively, u_s is the n_s-dimensional substructure configuration vector, η_s the n_s-dimensional force vector representing the forces exerted by all adjacent substructures on substructure s, and λ is an eigenvalue of the whole structure. The vector u_s generally contains both translational and angular displacements. The mass matrix m_s is positive definite and the stiffness matrix k_s is either positive definite or only positive semidefinite, depending on the boundary conditions at the external boundary of the substructure. Note that we distinguish between an external boundary S_{Es} and an internal boundary S_{rs} of substructure s. The external boundary S_{Es} is the physical substructure boundary and it coincides with the boundary of the whole structure, whereas the internal boundary S_{rs} is the boundary between substructure s and any adjacent substructure r $(r=1,2,...,m;$ $r\neq s)$. The boundary conditions are automatically taken into account in the discrete formulation (10). The force vector η_s reflects the effects of adjacent substructures on substructure s, where for each adjacent substructure r and internal boundary S_{rs} we write

$$(\eta_s)_{Ir} = \eta_{rs}, \qquad r,s=1,2,..,m;\ r\neq s \tag{11}$$

In Eq. (11), $(\eta_s)_{Ir}$ denotes the entries in the vector η_s corresponding to the generalized coordinates at the internal boundary points $P\epsilon S_{rs}$ and the vector η_{rs} is an unknown vector representing the force exerted by substructure r on the internal boundary S_{rs} of substructure s. It is assumed that the vectors $(\eta_s)_{Ir}$ and η_{rs} are m_{rs}-dimensional.

If the stiffness matrix k_s is positive definite, then it is also nonsingular and Eq. (10) admits the unique solution

$$u_s = f_s + \lambda A_s u_s \tag{12}$$

where

$$f_s = k_s^{-1}\eta_s = \sum_{\substack{r=1 \\ r\neq s}}^{m} (k_s^{-1})_{Ir}\eta_{rs} \tag{13a}$$

$$A_s = k_s^{-1} m_s \tag{13b}$$

In Eq. (13a), $(k_s^{-1})_{Ir}$ denotes an $n_s \times m_{rs}$ rectangular matrix formed by the columns of k_s^{-1} that multiply the generalized coordinates corresponding to the internal boundary points $P\epsilon S_{rs}$. Note that Eq. (12) is a reciprocal formulation of Eq. (10) and it forms the basis for the iterative procedure described in Sec. V for generating improved substructure admissible vectors.

If k_s is only positive semidefinite, then it is singular and Eq. (10) does not admit a unique solution. A characteristic of a positive semidefinite substructure is that it admits a number of independent solutions, known as the substructure rigid body modes, which satisfy $k_s u_s = 0$. As a result, all solutions that differ from a specific solution u_s of Eq. (10) by the addition of a linear combination of the rigid body modes are also solutions. Assuming that there are n_{Rs} rigid body modes for substructure s and denoting them by $u_s^{(i)}$ $(i=1,2,...,n_{Rs})$, the singularity of the stiffness matrix k_s is removed (in the same way as for a single structure) by requiring that the substructure configuration vector satisfy the n_{Rs} independent constraint equations

$$C_s u_s = 0 \tag{14}$$

where C_s is an $n_{Rs} \times n_s$ rectangular matrix. One simple choice of constraints is that rendering n_{Rs} entries in the vector u_s corresponding to translations and rotations of a single point equal to zero. The constraint equation (14) can be used to write a relation between an $(n_s - n_{Rs})$-dimensional vector y_s of independent generalized coordinates and the n_s-dimensional constrained vector u_{sc} in the form

$$u_{sc} = C_{sc} y_s \tag{15}$$

where C_{sc} is an $n_s \times (n_s - n_{Rs})$ rectangular matrix. A solution of Eq. (10) consists of the unique solution u_{sc} relative to the constraints (14) plus an arbitrary linear combination of the rigid body modes. Substituting Eq. (15) into the left side of Eq. (10) and premultiplying the result by the matrix C_{sc}^T, we obtain the $(n_s - n_{Rs})$ algebraic equations

$$\bar{k}_s y_s = C_{sc}^T \eta_s + \lambda C_{sc}^T m_s u_s \tag{16}$$

where $\bar{k}_s = C_{sc}^T k_s C_{sc}$ is an $(n_s - n_{Rs}) \times (n_s - n_{Rs})$ positive definite symmetric matrix. Equation (16) admits a unique solution which, upon substitution into Eq. (15), yields the solution of Eq. (10) in the form of Eq. (12), where in this case

$$f_s = \sum_{i=1}^{n_{Rs}} a_{si} u_s^{(i)} + \sum_{\substack{r=1 \\ r \neq s}}^{m} [C_{sc}(\bar{k}_s)^{-1} C_{sc}^T]_{Ir} \eta_{rs} \tag{17a}$$

$$A_s = C_{sc}(\bar{k}_s)^{-1} C_{sc}^T m_s \tag{17b}$$

IV. Rayleigh's Quotient and the Intermediate Structure

A first step in substructure synthesis is to write the Rayleigh quotient for the whole structure, formed by joining together all m substructures. The quotient is obtained by premultiplying Eq. (10) for each substructure s $(s=1,2,...,m)$ by u_s^T, adding together all resulting m equations, and solving for λ in the form

$$\lambda = R = \frac{\sum_{s=1}^{M} u_s^T k_s u_s - \sum_{s=1}^{m} u_s^T \eta_s}{\sum_{s=1}^{m} u_s^T m_s u_s} \tag{18}$$

In view of Eq. (11), Eq. (18) can be rewritten as

$$\lambda = R = \frac{\sum_{s=1}^{m} u_s^T k_s u_s - \sum_{s=1}^{m} \sum_{r=s+1}^{m} [(u_s)_{Ir} - (u_r)_{Is}]^T \eta_{rs}}{\sum_{s=1}^{m} u_s^T m_s u_s} \tag{19}$$

where it has been recognized that the internal boundary points on S_{rs} are the same as those on S_{sr} and that $\eta_{rs} = -\eta_{sr}$. The latter fact is a result of interpreting the vector η_{rs} as the force exerted by substructure r on the internal boundary of substructure s. The force η_{sr} exerted by substructure s on substructure r must be equal in magnitude and must act in the opposite direction as the force η_{rs}, assuming that coordinate axes for each substructure have the same orientation.

The geometric compatibility conditions defining an assembled structure are $(u_s)_{Ir} - (u_r)_{Is} = 0$. An equivalent statement is to require that

$$\eta_{rs}^T [(u_s)_{Ir} - (u_r)_{Is}] = 0, \qquad r,s = 1,2,...,m; \; r \neq s \tag{20}$$

be satisfied for all possible force vectors η_{rs}. Satisfaction of Eqs. (20) insures that the second double summation in the numerator of Eq. (19) equals zero. In the discrete substructure synthesis method,[17,18] the m_{rs}-dimensional vector η_{rs} is approximated by the sum

$$\eta_{rs} = \sum_{i=1}^{M_{rs}} a_{rsi} g_{rsi}, \qquad r,s = 1,2,...,m \tag{21}$$

where a_{rsi} are unknown coefficients, g_{rsi} are linearly independent vectors, and $M_{rs} \leq m_{rs}$. Substituting Eq. (21) into Eq. (20), the compatibility condition (20) is satisfied if

$$g_{rsi}^T [(u_s)_{Ir} - (u_r)_{Is}] = 0, \qquad r,s, = 1,2,...,m; \; r \neq s \tag{22}$$

holds for each vector g_{rsi} $(i=1,2,...,M_{rs})$. Equation (22) is recognized as approximating the geometric compatibility conditions and the vectors g_{rsi} are identified as weighting vectors.[17,18] The particular choice of weighting vectors and the number of weighting vectors used defines an intermediate structure. The weighting vectors serve as the basis vectors for a series expansion of the force vector acting at the internal boundary S_{rs} between two adjacent substructures r and s. The series expansion (21) is exact for the intermediate structure, although it is only approximate for the actual structure $(M_{rs} \leq m_{rs})$. The concept of an intermediate structure is discussed in detail in Refs. 17 and 18. All further discussion is directed toward the intermediate structure.

Let us now reconsider formulation (12) for each substructure. The vector f_s appearing in Eq. (12) and defined by Eq. (13a) or (17a) represents the substructure static response resulting fron conditions imposed on the internal boundaries S_{rs} $(r,s=1,2,...,m;\; r \neq s)$. For known vectors η_{rs}, which implies known vectors η_s, the static solution f_s can be obtained directly by solving $k_s f_s = \eta_s$, very likely by Gaussian elimination. In the case of an intermediate structure, the vectors are given in the form of series expansions in terms of known vectors multiplied by undetermined coefficients. Substituting the series (21) into Eqs. (13a) and (17a), we obtain the vectors f_s, respectively, as follows:

$$f_s = \sum_{\substack{r=1 \\ r \neq s}}^{m} \sum_{i=1}^{M_{rs}} a_{rsi} (k_s^{-1})_{Ir} g_{rsi} = \sum_{i=1}^{M_{sc}} a_{si} F_{si} \tag{23a}$$

$$f_s = \sum_{i=1}^{n_{Rs}} a_{si} u_s^{(i)} + \sum_{\substack{r=1 \\ r \neq s}}^{m} \sum_{i=1}^{M_{rs}} a_{rsi} [C_{sc}(\bar{k}_s)^{-1} C_{sc}^T]_{Ir} g_{rsi} = \sum_{i=1}^{M_{sc}} a_{si} F_{si} \tag{23b}$$

Equation (23a) is valid for positive definite substructures in which

$$M_{sc} = \sum_{\substack{r=1 \\ r \neq s}}^{m} M_{rs}$$

is the total number of weighting vectors associated with substructure s, a_{si} $(i=1,2,...,M_{sc})$ are unknown coefficients, and F_{si} $(i=1,2,...,M_{sc})$ are vectors defined as the static response of substructure s resulting from the application of forces on its internal boundary points. Similarly, Eq. (23b) is valid for positive semidefinite substructures in which

$$M_{sc} = n_{Rs} + \sum_{\substack{r=1 \\ r \neq s}}^{m} M_{rs}$$

Note that the rigid body modes are taken as the first n_{Rs} static solutions F_{si} $(i=1,2,...,n_{Rs})$ in Eq. (23b).

V. Subspace Iteration Based on Substructure Synthesis

By definition, the intermediate structure is a structure subject to only approximate geometric compatibility conditions, where the approximate compatibility is enforced by satisfying Eq. (22) for each internal boundary weighting vector g_{rsi} $(i=1,2,...,M_{rs})$. To define subspace iteration for the intermediate structure, it is necessary to produce either explicitly or implicitly the matrix operator $A = k^{-1} m$ associated with the assembled intermediate structure, where m and k are the intermediate structure mass and stiffness matrices, respectively. To produce the operator A, the reciprocal formulation (12) for each substructure must be considered. In view of Eqs. (23), Eq. (12) for each substructure contains a number of undetermined coefficients a_{si}

$(i=1,2,\ldots,M_{sc})$. An explicit matrix operator A for the intermediate structure can be obtained by substituting Eq. (12) for each substructure s into the approximate compatibility conditions (22) and solving the result for the coefficients a_{si} (see Ref. 20). The explicit operator $A=k^{-1}m$ can be used in subspace iteration to generate improved vectors for the assembled intermediate structure. But the same effect can be obtained by interchanging the order of satisfying the compatibility conditions and generating improved vectors, i.e., by using the substructure operator A_s in Eq. (12) first to generate improved substructure vectors and then determining the unknown coefficients a_{si} by satisfying the compatibility conditions. Of course, the process of satisfying the compatibility conditions (22) is also a part of the substructure synthesis method of Refs. 17 and 18. Moreover, the substructure synthesis method constitutes a Rayleigh-Ritz method for the intermediate structure. This leads us to consider an iterative procedure based on the substructure synthesis method in which the appropriate values of the coefficients a_{si}, i.e., the appropriate linear combinations of the static vectors F_{si} are determined automatically as a result of the synthesis. Such an iterative procedure can be thought of as implicitly producing a matrix operator A for an assembled intermediate structure. Thus, the convergence properties of such a procedure are similar to those for the subspace iteration and no new convergence proof or error analysis is needed. The advantage of such a procedure is that it allows improved vectors for each substructure to be produced independently of all other substructures. Note that, when using the present choice of admissible vectors and considering two adjacent substructures r and s, we know a priori that the unknown coefficients a_{rsi} are the negative of the corresponding coefficients a_{sri} $(i=1,2,\ldots,M_{rs})$, although this fact need not be reflected explicitly in the substructure synthesis algorithm for satisfying the compatibility conditions.

We are concerned with an iterative procedure for producing the solution to the eigenvalue problem for a general intermediate structure. We shall consider a positive semidefinite intermediate structure and denote the total number of rigid body modes by n_R. When the intermediate structure is positive definite, $n_R=0$. Hence, this procedure is valid for either a positive definite or a positive semidefinite intermediate structure.

Let us consider iterating simultaneously to the first q nonzero eigenvalues and associated eigenvectors for an intermediate structure. Any zero eigenvalues are associated with known rigid body modes. In the discrete substructure synthesis method,[17,18] the part of an intermediate structure eigenvector u_s which is in a substructure s $(s=1,2,\ldots,m)$ is represented in terms of N_s linearly independent substructure admissible vectors ϕ_{si} $(i=1,2,\ldots,N_s)$. The iterative procedure is defined by choosing specific admissible vectors ϕ_{si}^p at each iteration p, where the specific vectors chosen are based on Eq. (12) for each substructure. Of course, Eq. (12) must be considered in conjunction with Eqs. (23). We point out that if the intermediate structure is positive definite, some disjoint substructures are positive definite and others may be only positive semidefinite. On the other hand, if the intermediate structure is positive semidefinite all disjoint substructures are positive semidefinite. In view of Eqs. (12) and (23), we represent each substructure s at iteration p by the sum of $N_s=M_{sc}+q$ substructure admissible vectors.

$$u_{s,p}=\sum_{i=1}^{N_s}\zeta_{si}\phi_{si}^p=\sum_{i=1}^{M_{sc}}\zeta_{si}F_{si}+\sum_{i=M_{sc}+1}^{N_s}\zeta_{si}\phi_{si}^p \tag{24}$$

where ζ_{si} $(i=1,2,\ldots,N_s)$ are unknown coefficients to be determined by the synthesis. Note that the first M_{sc} admissible vectors in Eq. (24) are taken as the static response vectors F_{si} $(i=1,2,\ldots,M_{sc})$ defined in Eqs. (23). If the substructure is positive semidefinite, n_{Rs} substructure rigid body modes are included as static vectors F_{si} [see Eq. (23b)]. The remaining q admissible vectors are taken during the initial iteration, step $p=0$, to be arbitrary, linearly independent, substructure admissible vectors that are independent of the static vectors F_{si} $(i=1,2,\ldots,M_{sc})$. For simplicity, q is taken to be the same for all substructures. The specific choice of the remaining q admissible vectors during subsequent steps $p=1,2,\ldots$ will be discussed shortly.

At each iteration step $p=0,1,\ldots$, an approximate intermediate structure eigensolution is produced by the discrete substructure synthesis method, in which each substructure is represented by th sum (24), which can be written conveniently in the matrix form

$$u_{s,p}=\Phi_s^p\zeta_s \tag{25}$$

where Φ_s^p is an $n_s\times N_s$ matrix of admissible vectors and ζ_s is the N_s-dimensional vector of coefficients. The substructure synthesis method consists of first formulating the disjoint Rayleigh's quotient

$$R_d=\zeta_d^Tk_d\zeta_d/\zeta_d^Tm_d\zeta_d \tag{26}$$

where $\zeta_d=\{\zeta_1^T\zeta_2^T\cdots\zeta_m^T\}^T$ is an N-dimensional disjoint configuration vector,

$$N=\sum_{s=1}^{m}N_s,$$

and k_d and m_d are the $N\times N$ block diagonal matrices

$$k_d=\text{block-diag }\Phi_s^{pT}k_s\Phi_s^p,\qquad s=1,2,\ldots,m \tag{27a}$$

$$m_d=\text{block-diag }\Phi_s^{pT}m_s\Phi_s^p,\qquad s=1,2,\ldots,m \tag{27b}$$

Rayleigh's quotient for the intermediate structure is obtained by requiring that a total of M_c geometric compatibility conditions of the form of Eq. (22) be satisfied (see Refs. 17 and 18). The M_c compatibility conditions are used as constraint equations, leading to a relation between the disjoint vector ζ_d and the n-dimensional $(n=N-M_c)$ unconstrained vector ζ in the form

$$\zeta_d=C\zeta \tag{28}$$

where C is an $N\times n$ rectangular matrix. We distinguish between the number n of degrees of freedom for the reduced intermediate structure and the total number n_T of degrees of freedom of the actual structure, where $n\ll n_T$. The coefficient vectors ζ_s for all substructures s $(s=1,2,\ldots,m)$ are determined by rendering the Rayleigh quotient for an intermediate structure stationary, which amounts to solving the n-dimensional algebraic eigenvalue problem

$$K\zeta=\Lambda^nM\zeta \tag{29}$$

where $K=C^Tk_dC$ and $M=C^Tm_dC$. The algebraic eigensolution consists of n computed eigenvalues Λ_r^n $(r=1,2,\ldots,n)$, approximating the true intermediate structure eigenvalues. Corresponding to each eigenvalue Λ_r^n there are values of the coefficients $\zeta_{si}^{(r)}$ which, when substituted into Eq. (24) for each substructure s, yield that part of the computed eigenvector within substructure s, namely $u_{s,p}^{(r)}$.

The computed eigenvalues Λ_r^n provide upper bounds for the first n true intermediate structure eigenvalues.[17,18] The accuracy of the bounds depends on the particular choices of substructure admissible vectors ϕ_{si}^p $(i=1,2,\ldots,N_s)$ as well as their numbers N_s $(s=1,2,\ldots,m)$. Consistent with the concept of subspace iteration, the accuracy can be improved by producing better sets of admissible vectors to represent each substructure without increasing the number of vectors in each set. We are concerned with improving the accuracy of the first q nonzero eigenvalues and associated eigenvectors. To this end, based on Eq. (12) for each substructure s, we utilize the q

improved substructure admissible vectors

$$\phi^{p+1}_{s,M_{sc}+i} = A_s u^{(n_R+i)}_{s,p}, \qquad i=1,2,\ldots,q \tag{30}$$

where the matrix operator A_s is given by either Eq. (13b) or Eq. (17b), depending on whether substructure s is positive definite or positive semidefinite. The vectors $u^{(n_R+i)}_{s,p}$ $(i=1,2,\ldots,q)$ in Eq. (30) are computed eigenvectors determined in the preceding substructure synthesis. The justification for using Eq. (30) to generate improved vectors is given in the first paragraph of this section and also in Ref. 20. The computations indicated by Eq. (30) for substructures s are independent of all other substructures. Hence, the improvement process for all substructures can be performed in parallel. Note that according to Eq. (30) only the last q substructure admissible vectors are changed in each iteration. The first N_s-q substructure admissible vectors must always be taken as the substructure static response vectors. Moreover, only the computed eigenvectors corresponding to the first q nonzero computed eigenvalues are used in Eq. (30). As mentioned earlier, if the intermediate structure eigenvalue problem is positive definite, then $n_R=0$ in Eq. (30).

The next iteration, step $p+1$ $(p=0,1,\ldots)$, consists of solving the algebraic eigenvalue problem obtained by substituting the improved admissible vectors (30) into Eq. (24) and using the resulting sets of substructure admissible vectors in the synthesis for the same intermediate structure. The entire procedure for a particular intermediate structure is summarized as follows:

1) Choose q trial substructure admissible vectors ϕ^0_{si} $(i=M_{sc}+1,\ldots,M_{sc}+q;\ s=1,2,\ldots,m)$.

2) Using static vectors F_{si} $(i=1,2,\ldots,M_{sc})$ and admissible vectors ϕ^p_{si} $(i=M_{sc}+1,\ldots,M_{sc}+q)$ in Eq. (24) to represent each substructure s, compute Λ^n_r and $u^{(r)}_{s,p}$ $(r=1,2,\ldots,n)$ by solving the reduced-order eigenvalue problem (29) obtained from the discrete substructure synthesis method.

3) If the computed eigenvalues and eigenvectors have converged, stop iterating. Otherwise, go to step 4.

4) Use Eq. (30) to generate q improved admissible vectors ϕ^{p+1}_{si} $(i=M_{sc}+1,\ldots,M_{sc}+q)$ for each substructure s $(s=1,2,\ldots,m)$ and go to step 2 with $p=p+1$.

Global admissible vectors, i.e., vectors encompassing generalized coordinates for the entire structure, are generated implicitly in the discrete substructure synthesis method, where a total of n global admissible vectors are generated.[17,18] The global admissible vectors are basis vectors spanning an n-dimensional vector space. Because the parts of q computed eigenvectors within a substructure s are used in Eq. (30) for each substructure s $(s=1,2,\ldots,m)$, as the number of iterations becomes infinite, the n-dimensional vector space contains the q-dimensional subspace spanned by the true eigenvectors corresponding to the first q nonzero true eigenvalues of the intermediate structure. In this way, the proposed iteration procedure converges to the first q nonzero true eigenvalues and associated eigenvectors of the intermediate structure. The substructure synthesis method, however, always produces n computed eigenvalues and eigenvectors approximating the intermediate structure eigenvalues and eigenvectors. Whereas the first n_R+q eigenvalues and associated eigenvectors converge to true eigenvalues and eigenvectors, the remaining $n-n_R-q$ eigenvalues and associated eigenvectors may converge to quantities other than true eigenvalues and eigenvectors. They do, nevertheless, approximate the higher $n-n_R-q$ true eigenvalues and associated eigenvectors, where the computed eigenvalues are always upper bounds to the true eigenvalues of the intermediate structure.

In the above discussion, Eq. (30) in terms of the matrix operator A_s is used to calculate improved admissible vectors. A matrix operator A_s, defined by Eq. (13b) or Eq. (17b), is obtained by inverting a stiffness matrix k_s or a constrained stiffness matrix $\bar{k}_s$, respectively. In practice, rather than invert the substructure stiffness matrix k_s to obtain Eq. (30), it is more expedient computationally to solve the n_s simultaneous algebraic equations

$$k_s \phi^{p+1}_{s,M_{sc}+i} = m_s u^{(n_R+i)}_{s,p}, \qquad i=1,2,\ldots,q \tag{31}$$

perhaps by Gaussian elimination. In the case of a positive semidefinite substructure, rather than invert the constrained stiffness matrix $\bar{k}_s$ to obtain Eq. (30), it is more expedient computationally to solve the n_s-n_{Rs} simultaneous algebraic equations

$$\bar{k}_s y^{p+1}_{si} = C^T_{sc} m_s u^{(n_R+i)}_{s,p}, \qquad i=1,2,\ldots,q \tag{32a}$$

and substitute the results into

$$\phi^{p+1}_{s,M_{sc}+i} = C_{sc} y^{p+1}_{si}, \qquad i=1,2,\ldots,q \tag{32b}$$

The computational effort involved in solving the algebraic equations [(31) or (32)] is relatively small compared to that in solving Eq. (4) for the whole structure, even if the number n_s of substructure degrees of freedom is large. Hence, for structures composed of discrete substructures this iterative procedure is a practical computational tool.

VI. Numerical Example

Let us consider the same discrete structure model as that considered in Refs. 17 and 18. An approximate eigensolution for three different intermediate structures representing the discrete structure was obtained in Refs. 17 and 18. The same intermediate structures are considered in the present example. The difference is that here we converge to the eigensolution for each intermediate structure by using the subspace iteration based on substructure synthesis to improve a given number of admissible vectors, rather than increasing the number of admissible vectors used.

The discrete structure is in the form of an unrestrained framework (Fig. 1) composed of two identical substructures. Each substructure (Fig. 2) is modeled by the finite-element method, where the model consists of 34 elements and 29 unrestrained nodal points. All members in a substructure are assumed to be slender aluminum tubes with the same dimensions and wall thicknesses. For simplicity, we assume that each element is capable of bending displacements in the z-direction and torsion. This implies that at each nodal point there are three degrees of freedom: the translation in the z-direction, the rotation about the x-axis, and the rotation about the y-axis. The dimension of each substructure finite element model is $n_s=87$ $(s=1,2)$. In Refs. 17 and 18, a substructure was represented by a truncated set of admissible vectors obtained by discretizing admissible functions of a thin rectangular plate that is free along all its edges. Considering substructure 1, admissible functions ϕ_i (D) can be taken in the form of products of the functions $\chi_i(x)=(x/L_x)^i$ and $\psi_j(y)=(y/L_y)^j$ as follows:

$$\phi_i(D) = [1, \chi_1(x), \psi_1(y), \chi_2(x), \chi_1(x)\psi_1(y), \psi_2(y),$$
$$\chi_3(x), \chi_2(x)\psi_1(y), \chi_1(x)\psi_2(y), \psi_3(y), \ldots] \tag{33}$$

The first three of these functions represent rigid body modes of the free plate and result in rigid body modes for the discrete substructure. The finite element model of substructure 1 is constructed in such a way that the rth, $(r+1)$st, and $(r+2)$nd entries in the configuration vector u_1 are the translation in the z-direction, the rotation about the x-axis, and the rotation about the y-axis at node j, respectively, where $r=3(j-1)+1$ $(j=1,2,\ldots,29)$. Hence, substructure admissible vectors ϕ_{1i} are constructed by taking values of the function ϕ_i and its derivatives $\partial\phi_i/\partial y$ and $-\partial\phi_i/\partial x$, evaluated at the point (x_r,y_r) in D as the rth, $(r+1)$st, and $(r+2)$nd entries in the 87-dimensional admissible vector ϕ_{1i}, respectively. Note that admissible vectors obtained by discretizing the functions (33) possess the general shapes associated with these functions. For substructure 2, admissible vectors can be obtained by discretizing the same functions (33) with x replaced by $(-x)$ and y replaced by $(-y)$.

The internal boundary S_{12} between the substructures consists of five nodal points equally spaced along the line $x=0$, $-L_y/2 \le y \le L_y/2$. The total number of compatibility conditions connecting the two substructures is then $m_{12}=15$. As in Refs. 17 and 18, we satisfy the compatibility conditions approximately by taking internal boundary weighting vectors g_{12i} $(i=1,2,...,M_{12})$ in the form of standard unit vectors and consider the three different intermediate structures defined by: 1) using five weighting vectors so as to satisfy compatibility exactly at the nodal point in the center of the internal boundary and to enforce equality of displacements at the two exterior points, 2) using seven weighting vectors so as to enforce equality of rotations about the y-axis at the two exterior points on the internal boundary in addition to the previous five conditions, and 3) using nine weighting vectors so as to satisfy compatibility exactly at the center and the two exterior nodal points on the internal boundary.

The actual discrete structure is unrestrained, i.e., it is only positive semidefinite. Hence, both disjoint substructures are only positive semidefinite and they both possess $n_{R1}=n_{R2}=3$ rigid body modes. The rigid body modes for both substructures are the vectors generated by discretizing the first three functions specified in Eq. (33). A characteristic of a positive semidefinite substructure is that its stiffness matrix k_s $(s=1,2)$ is singular. For each substructure, we shall artificially remove the singularities by imposing three constraints at the nodal point in the center of the internal boundary as follows: 1) the displacement in the z-direction is equal to zero, 2) the rotation about the x-axis is equal to zero, and 3) the rotation about the y-axis is equal to zero. In view of these artificial constraints, the three weighting vectors which serve to enforce compatibility at the nodal point in the center of the internal boundary are interpreted as forces and torques applied at a fixed point. Forces and torques applied at a fixed point produce only trivial substructure static response vectors. We shall discard the trivial static vectors and consider that to each one of these three weighting vectors there corresponds a substructure static response vector in the form of one of the substructure rigid body modes. The remaining weighting vectors for each intermediate structure are interpreted as either a unit force or a unit torque applied at an internal boundary point and they produce nontrivial substructure static response vectors, where the response is calculated relative to the artificially constrained nodal point.

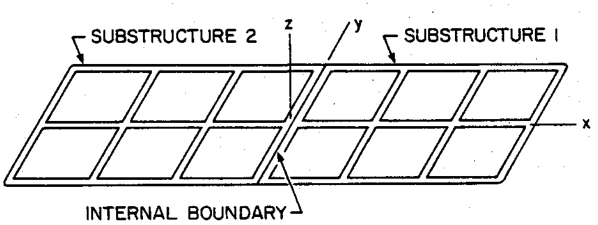

Fig. 1 The example structure.

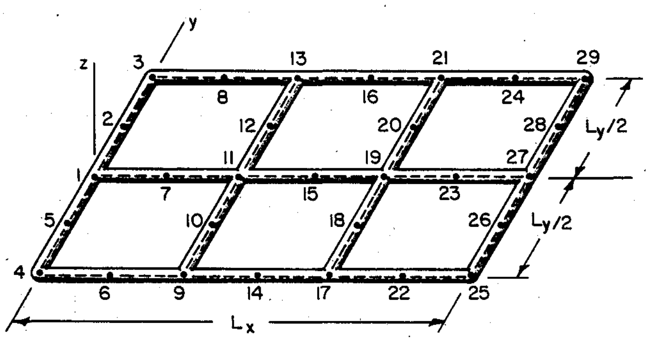

Fig. 2 A single substructure.

In view of the above discussion, in the synthesis for the intermediate structures defined by using $M_{12}=5$, 7, or 9 weighting vectors, there are $M_{12}=5$, 7, or 9 static response vectors for each substructure, respectively. Consistent with the iterative procedure developed in Sec. V, the first M_{12} admissible vectors used to represent each substructure are taken as static vectors. Note that in all cases the first three static vectors coincide with substructure rigid body modes. We shall consider iterating to the first $q=7$ nonzero eigenvalues and associated eigenvectors for each intermediate structure. Hence, in the synthesis for the intermediate structures defined by $M_{12}=5$, 7, and 9 weighting vectors, we represent each substructure by a total of $N_1=N_2=M_{12}+7=12$, 14, and 16 admissible vectors, respectively. In the initial iteration step, i.e., the step corresponding to $p=0$, the last seven admissible vectors representing substructure 1 are taken to be the vectors generated by discretizing or sampling in space the fourth function through the tenth function in Eq. (33), respectively. For substructure 2 in the step $p=0$, the last seven admissible vectors are taken to be the vectors generated by discretizing or sampling in space the same functions in Eq. (33) with x replaced by $(-x)$ and y replaced by $(-y)$.

The first eight nonzero computed eigenvalues for each intermediate structure are tabulated in Table 1 for four iterations corresponding to $p=0$, 1, 2, and 3. In addition, three zero eigenvalues associated with the intermediate structure rigid body modes are obtained in all cases. The first eight nonzero "exact" eigenvalues of the actual discrete structure, obtained by solving the eigenvalue problem associated with the finite element model of order 159, are displayed in the last column of Table 1. The exact eigenvalues were obtained by first reducing the eigenvalue problem by means of similarity transformations to that of a tridiagonal matrix and then computing the eigenvalues by the QL method. Note, that in the initial computations, i.e., in the iteration corresponding to $p=0$, the results are comparable to those presented in Refs. 17 and 18. In particular, there is fairly good agreement between the computed eigenvalues obtained by using the initial admissible vectors and the exact eigenvalues. The agreement is generally not as good as that observed in Refs. 17 and 18, but this is because fewer substructure admissible vectors are used here.

At the end of the first iteration, the step corresponding to $p=1$, the first seven nonzero eigenvalues computed using refined substructure admissible vectors are very nearly exact for each particular intermediate structure. In fact, none of the first seven nonzero computed eigenvalues for any of the three intermediate structures is changed by more than 0.15% after two subsequent iterations. This suggests that the first iteration is the most important one in obtaining a highly accurate set of refined admissible vectors to represent each substructure. Note that as the number of iterations increases the first seven nonzero computed eigenvalues for each intermediate structure decrease, or at least do not increase, and they converge to the actual eigenvalues of each intermediate structure. This is because the iteration is on $q=7$ intermediate structure eigenvectors and because the substructure synthesis method always produces computed eigenvalues which are the upper bounds to the eigenvalues of a particular intermediate structure. Such a statement cannot be made about computed eigenvalues higher than the seventh nonzero eigenvalue. For instance, we observe that as the number of iterations increases, the eighth nonzero eigenvalue for each intermediate structure actually increases. However, although they do not converge to actual intermediate structure eigenvalues, the nonzero computed eigenvalues higher than the seventh one remain upper bounds to the eigenvalues of a particular intermediate structure.

Table 1 Computed discrete structure eigenvalues using subspace iteration

No. of weighting vectors	No. of the computed eigenvalue	Initial substructure synthesis	First iteration ($p=1$)	Second iteration ($p=2$)	Third iteration ($p=3$)	"Exact" eigenvalues
5	4	1.377387	1.070465	1.070465	1.070465	1.393140
	5	5.386344	4.779372	4.779370	4.779370	4.904380
	6	14.190652	11.211618	11.211321	11.211320	11.211690
	7	21.052184	17.671796	17.670745	17.670745	19.206543
	8	67.786763	31.517985	31.476690	31.476309	38.635709
	9	85.767091	55.742722	55.676725	55.676664	56.005250
	10	143.457039	90.464662	90.423158	90.422984	111.403770
	11	190.609284	113.777058	115.940260	116.066436	112.537891
7	4	1.396853	1.393140	1.393140	1.393140	1.393140
	5	5.244594	4.779371	4.779370	4.779370	4.904380
	6	14.188884	11.211507	11.211320	11.211320	11.211690
	7	21.751966	19.206826	19.206543	19.206543	19.206543
	8	68.663808	38.664270	38.635780	38.635709	38.635709
	9	84.709070	55.711457	55.676707	55.676664	56.005250
	10	187.801421	111.567301	111.403968	111.403771	111.403770
	11	190.119219	113.173465	114.322408	114.380181	112.537891
9	4	1.396825	1.393140	1.393140	1.393140	1.393140
	5	5.091575	4.904380	4.904380	4.904380	4.904380
	6	14.170942	11.211898	11.211690	11.211690	11.211690
	7	20.617433	19.206619	19.206543	19.206543	19.206543
	8	68.346635	38.673786	38.636926	38.635750	38.635709
	9	84.948773	56.016369	56.005261	56.005250	56.005250
	10	175.830800	111.465477	111.404060	111.403772	111.403770
	11	176.801999	113.101137	113.988430	114.078850	112.537891

Let us also note that the intermediate structure defined by using five unit weighting vectors yields computed eigenvalues which are lower than the exact values. The computed eigenvalues tend to increase as additional internal boundary weighting vectors are used to satisfy geometric compatibility between the substructures more accurately. The largest increase in the eigenvalues is observed when seven rather than five weighting vectors are used, and only very small additional increases are observed when the number of weighting vectors is increased to nine. This corroborates a statement made in Refs. 17 and 18 that rapid convergence of the intermediate structures to the actual structure occurs. Seven unit weighting vectors are sufficient to insure very accurate compatibility for the purpose of computing the lower eigenvectors, at least in the case of this example structure. The intermediate structure defined by using nine weighting vectors yields computed eigenvalues after the first iteration which are virtually exact for the actual structure. Note that in this example, as the number of weighting vectors is increased, the number of substructure admissible vectors is also increased because additional substructure static solution vectors are considered. This explains why several of the computed eigenvalues displayed in Table 1 actually decrease slightly as the number of weighting vectors is increased. In these cases, the increase in the computed eigenvalues attributed to the addition of several weighting vectors is counteracted by the larger decrease attributed to the addition of several admissible vectors.

VII. Summary

A subspace iteration procedure for structures composed of discrete substructures has been developed. The procedure permits obtaining the actual eigensolution for an intermediate structure. At each iteration, the calculations for each substructure are independent of those for all other substructures and they can be performed in parallel. The entire procedure for a particular intermediate structure has been summarized in Sec. V.

It must be pointed out that a different subspace iteration procedure for structures composed of substructures has been presented recently.[21] Because subspace iteration allows computing a progressively more accurate partial eigensolution for an assembled structure, it is suggested in Ref. 21 that subspace iteration with substructuring, in particular the iterative procedure of Ref. 21, supersedes the substructure synthesis method. The developments of this paper demonstrate that this is not the case. Indeed, the substructure synthesis method of Refs. 17 and 18 is an integral part of the present iterative procedure. In Ref. 21, the trial vectors for subspace iteration must be specified for the structure as a whole. A significant advantage of the present procedure is that trial admissible vectors chosen to represent each individual substructure are chosen independently of all other substructures. Moreover, whereas Ref. 21 considers only positive definite structures, the present procedure (incorporating the substructure synthesis concepts) is applicable to assembled structures that are either positive definite or positive semidefinite.

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